TaylorShift:

Shifting the Complexity of Self-Attention from Squared to Linear (and Back) using Taylor-Softmax

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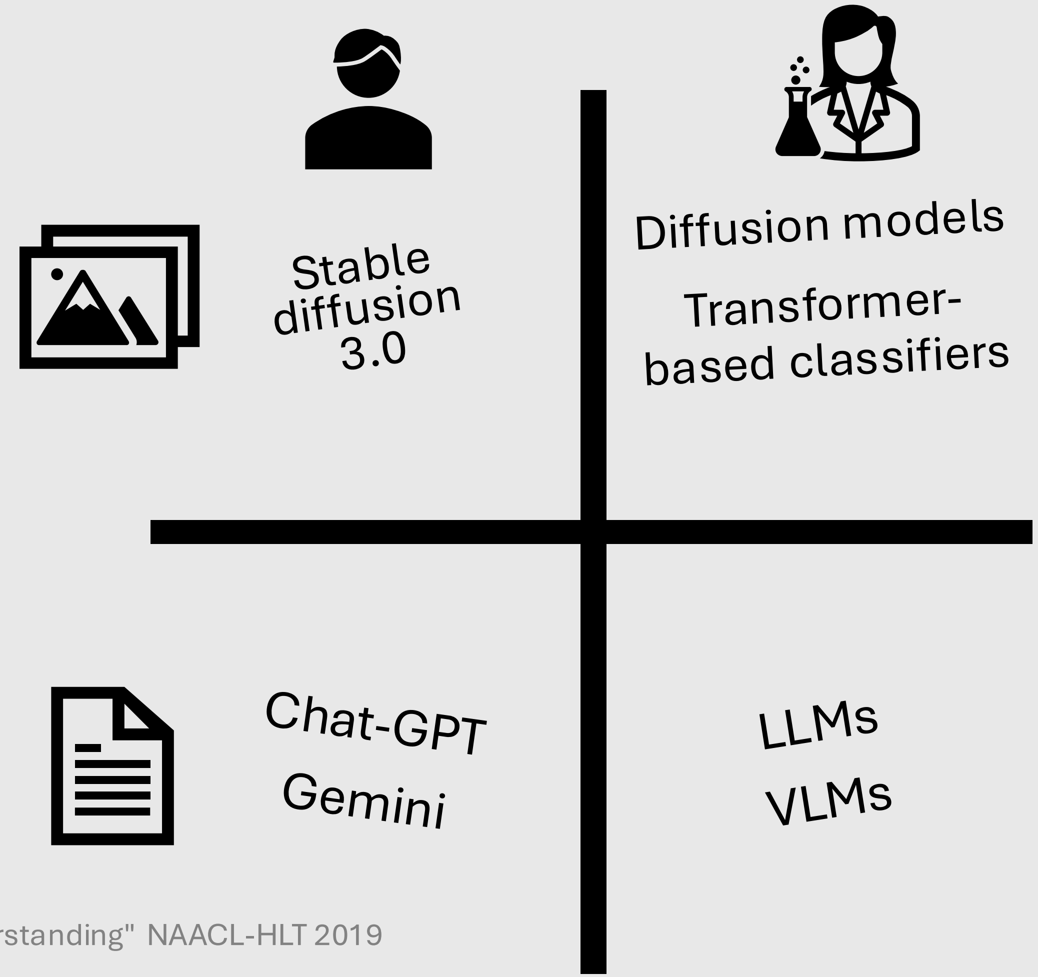
Modern AI is largely attention-based.

Introduction of Transformer models by Vaswani et al. [1]

BERT [2]

GPT-2 [3]

ViT [4]

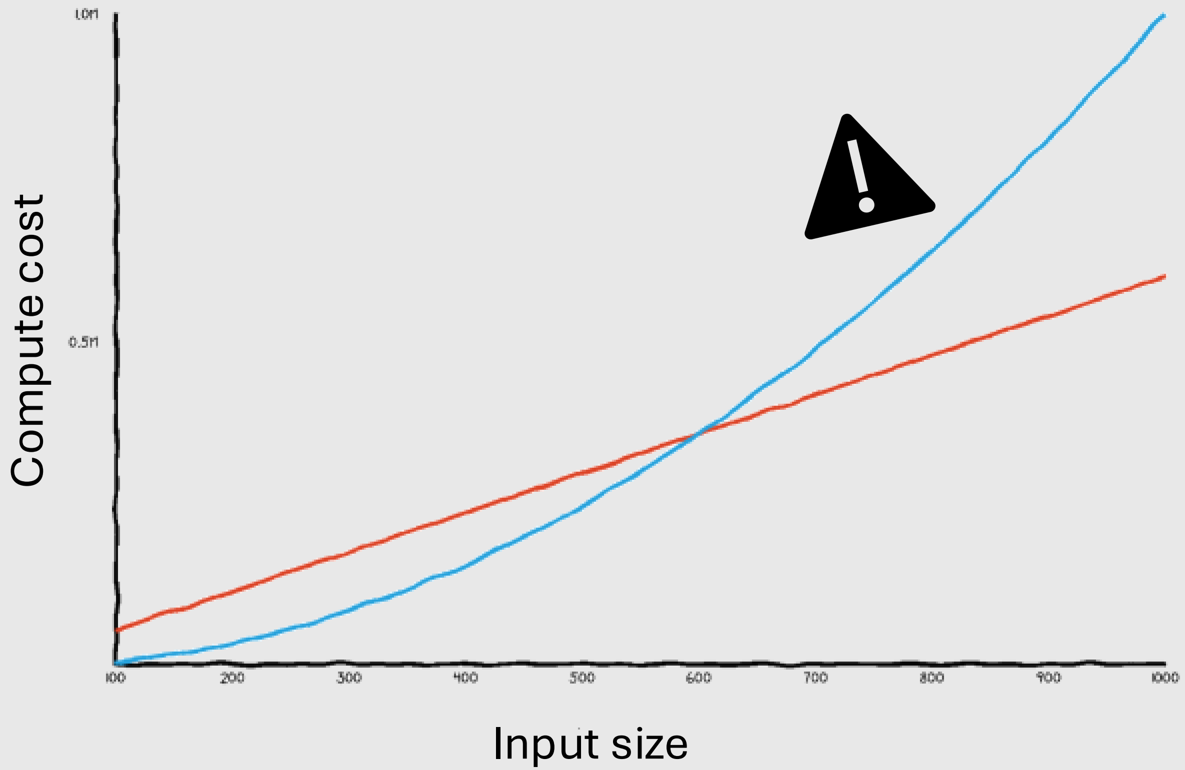


[1] Vaswani et al. "Attention Is All You Need" NeurIPS 2017
[2] Devlin et al. "BERT: Pre-training of Deep Bidirectional Transformers for Language Understanding" NAACL-HLT 2019
[3] Radford et al. "Language Models are Unsupervised Multitask Learners" 2019
[4] Dosovitskiy et al. "An Image is Worth 16x16 Words: Transformers for Image Recognition at Scale" ICLR 2021

Quadratic complexity prevents us from utilizing very long inputs.

Long inputs allow for:

- More context information
- (Updated) World knowledge



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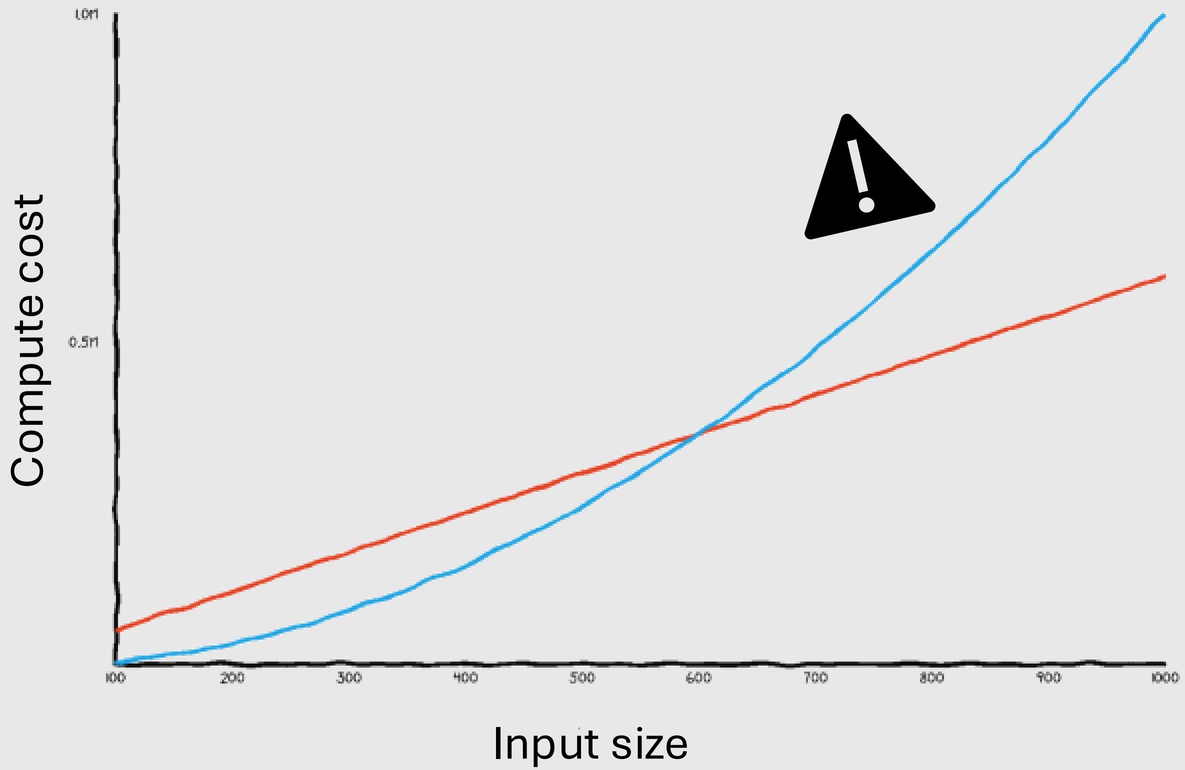
- High-resolution images
- More localized information



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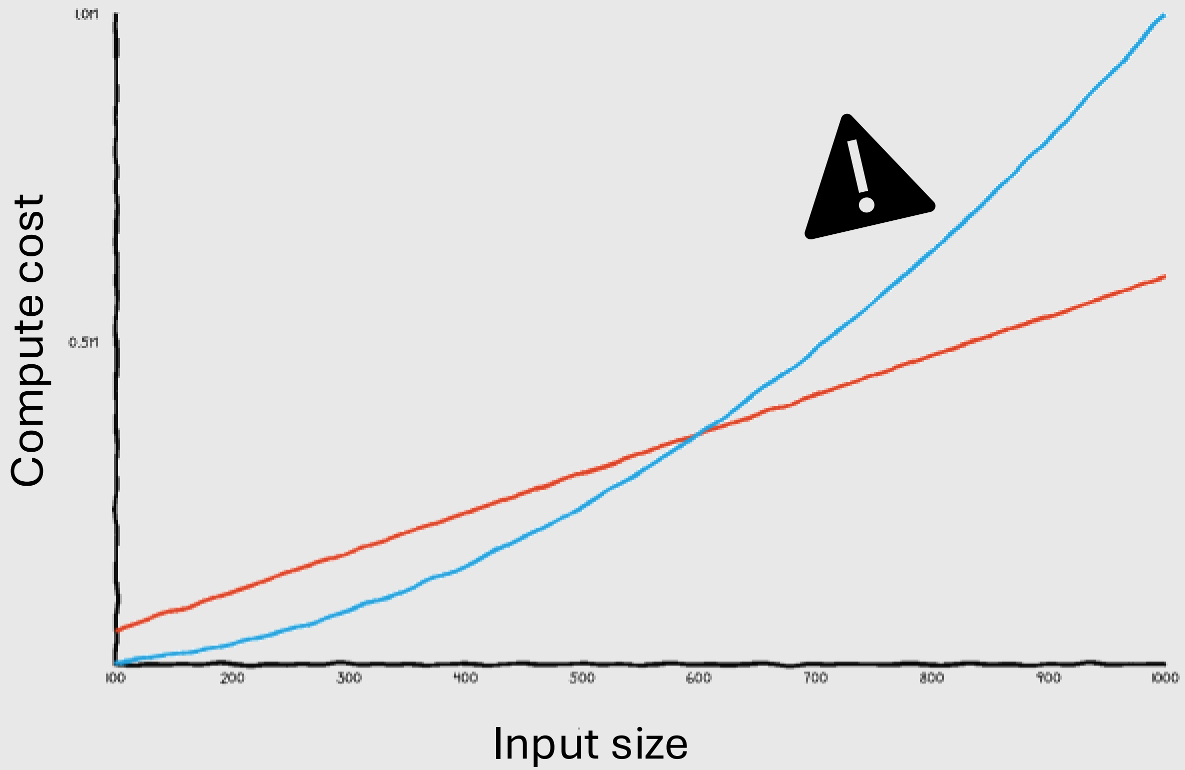
- Multiple modalities
- Concatenation of information



TaylorShift is attention using the Taylor series of the exponential.

	Attention	**Direct-TaylorShift**	
Activation	$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\approx \sum_{n=0}^{k} \frac{x^n}{n!}$	$k = 2$

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Interaction Function	$\text{softmax}(x) = \frac{\exp(x)}{\|\| \exp(x) \|\|_1}$	$\text{T-SM}^{(k)}(x) = \frac{\sum_{n=0}^{k} \frac{x^n}{n!}}{\left\|\left\| \sum_{n=0}^{k} \frac{x^n}{n!} \right\|\right\|_1}$

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Attention Calculation	$Y = \mathrm{softmax}(QK^\top)V$	$Y = \text{T-SM}^{(k)}(QK^\top)V$

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Attention Calculation	$Y = \mathrm{softmax}(QK^\top)V$	$Y = \text{T-SM}^{(k)}(QK^\top)V$ Direct-TaylorShift
Complexity $Q, K, V \in \mathbb{R}^{N \times d}$	$\mathcal{O}(N^2 d)$	$\mathcal{O}(N^2 d)$

We can reorder calculations for efficient calculation.

$$\text{T-SM}(QK^\top)V \;=\; \ell_1 - \text{norm}\left[\sum_{n=0}^{2} \frac{(QK^\top)^n}{n!}\right] V$$

1. Matrix multiplication (QK)
2. Activation function
3. Normalization
4. Matrix multiplication (KV) $\mathcal{O}(N^2 d)$

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Efficient-TaylorShift

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3. Matrix multiplication (QK)
4. Normalization $\mathcal{O}(N d^3)$

We counteract numerical instabilities by introducing multiple normalizations.

Normalization at the end

⇒ large intermediate results

⇒ numerical instability

Expr.	$(K^{\boxtimes 2})^\top V = A_{\text{mod}}$	$(QK^T)^2 V$	$QK^\top V$	Y_{denom}	Y
Size	$\frac{N+1}{\sqrt{d}}$	$\frac{N}{d}$	$\sqrt{N}\frac{4d+1}{4d}$	$N\frac{d+2}{2d}$	$\sqrt{\frac{d}{N}}$

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Expr.	$(K^{\boxtimes 2})^\top V = A_{\mathrm{mod}}$	$(QK^T)^2 V$	$QK^\top V$	Y_{denom}	Y
Size	$\frac{N+1}{N}$	1	$\frac{\sqrt{d}}{\sqrt{N}} + \frac{1}{4\sqrt{Nd}}$	$\frac{d+2}{2d}$	$\sqrt{\frac{d}{N}}$

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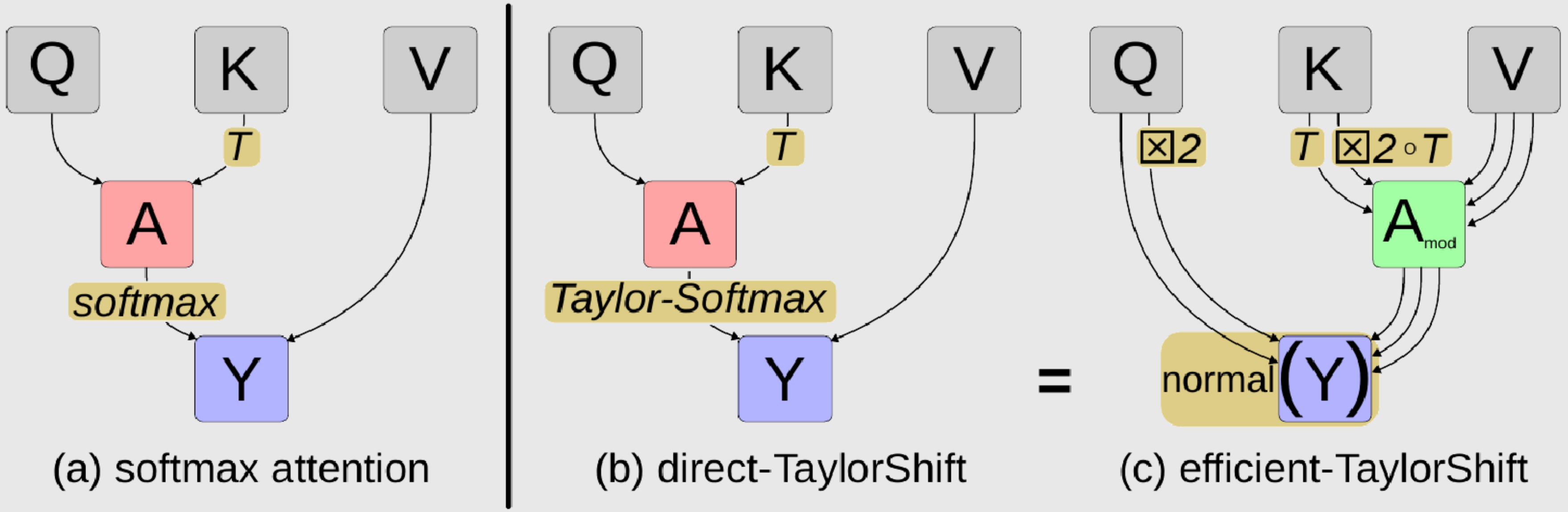
TaylorShift calculates token-to-token interactions in linear time.



1. Softmax → Taylor-Softmax
2. Reorder operations for efficient calculation
3. Normalize intermediate results

TaylorShift is efficient for long sequences, depending on d.

When is $O(N d^3)$ actually better than $O(N^2 d)$?

	Speed
Proxy Metric	# FLOPs

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d	8	16	32	64	128
N_0	73	273	1057	4161	16513
N_1	47	159	574	2174	8446

Increasing the number of attention-heads increases TaylorShift's efficiency.

Reducing the dimension d makes TaylorShift more efficient.

Reduces the representational capacity of the model.

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Change the number of attention-heads instead.

$$d = \frac{d_{\text{embed}}}{h} \qquad h \times \text{ops}_{\text{eff}} \qquad h \times \text{entr}_{\text{eff}}$$

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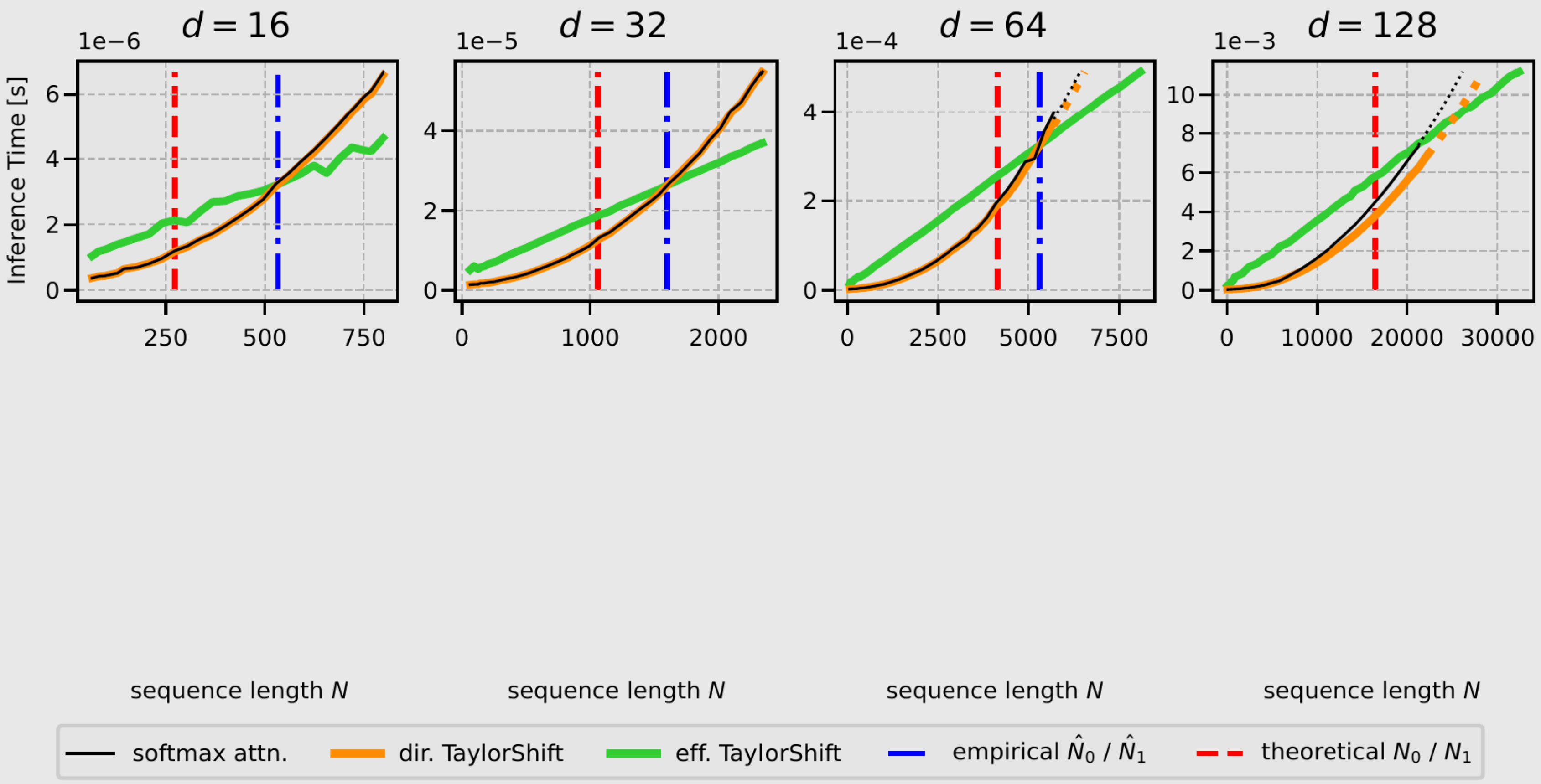
⇒ Increase efficiency without decreasing overall capacity by increasing the number of heads

TaylorShift is efficient for long sequences, as predicted.

Speed:
- Direct-TaylorShift outperforms attention
- Intersection point is larger than predicted
- Slow memory reads on current hardware



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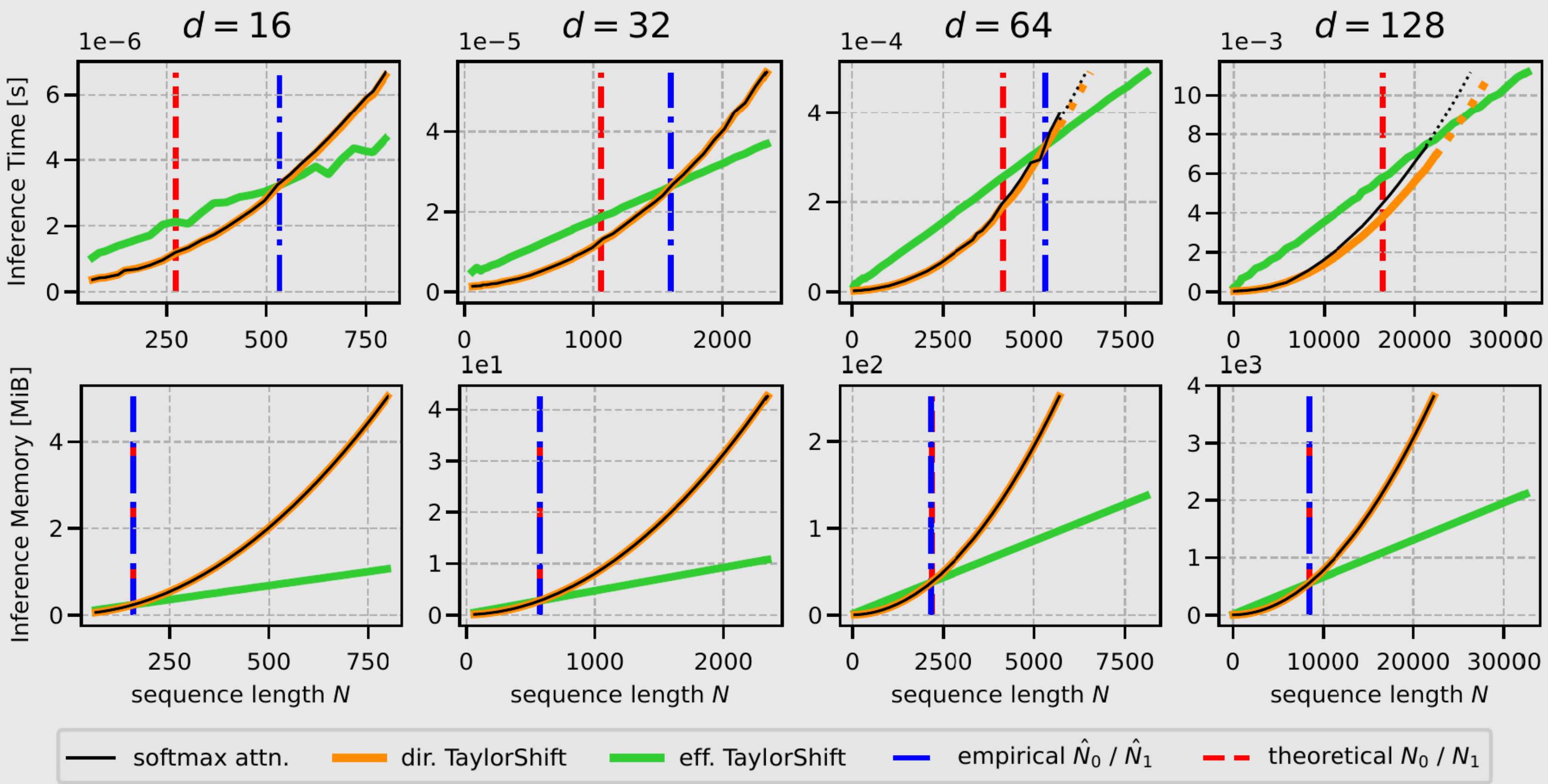
Memory:
- Almost exactly like predicted theoretically



TaylorShift outperforms standard attention in 4/5 tasks.

TaylorShift outperforms the other models, including the baseline Transformer encoder, in (at least) 4/5 tasks.

Model	CIFAR (Pixel)	IMDB (Byte)	ListOps	ImageNet (Ti)	ImageNet (S)	Average
Linformer	29.2	58.1	-	64.3	76.3	(57.0)
RFA	44.9	65.8	-	-	-	(55.4)
Performer	34.2*	65.6*	35.4*	62.0*	67.1*	52.9
Reformer	44.8	63.9	**47.6**	73.6	76.2*	61.2
Nystromformer	**49.4**	65.6	44.5	75.0	78.3*	62.6
EVA	46.1	64.0	45.3	73.4	78.2	61.4
Transformer	44.7	65.8	46.0	**75.6**	79.1	62.2
Ours	47.6	**66.2**	46.1	75.0	**79.3**	**62.8**

Increasing the number of heads speeds up TaylorShift and can increase accuracy.

Increasing the number of heads:
- Reduces memory consumption
- Increases speed
- Sometimes increases accuracy

h	d	Acc [%]	direct TP [ims/s]	direct Mem [MiB@16]	efficient TP [ims/s]	efficient Mem [MiB@16]
4	64	47.1	12 060	596	2 975	840
8	32	47.5	7 657	1 111	5 749	585
16	16	47.3	4 341	2 135	9 713	459
32	8	46.9	2 528	4 187	14 087	397
64	4	45.9	1 235	8 291	13 480	125

Accuracy, throughput and VRAM usage of TaylorShift on the CIFAR Pixel task with increased number of attention heads with:

$d_{\text{embed}} = d \times h = 256$ $\qquad N = 1024$

TaylorShift:
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Website

PDF

Code